AMENDMENTS
In the Specification

[0031] This particular problem motivated us to introduce the radial frame multiresolution analysis. Our construction is based on a very general multiresolution scheme of abstract Hilbert spaces developed by Papadakis in ~~[16]~~ M. Papadakis. Generalized Frame Multiresolution Analysis of Abstract Hilbert Spaces. 2001, namely the Generalized Frame Multiresolution Analysis (GFMRA). The main characteristic of GFMRAs is that they can be generated by redundant sets of frame scaling functions. In fact, GFMRAs encompass all classical MRAs in one and multidimensions as well as the FMRAs of Benedetto and Li (~~[6]~~ see J.J. Benedetto and S. Li. The Theory of Multiresolution Analysis Frames and Applications to Filter Banks. *Appl. Comp. Harm. Anal.,* 5:389-427, 1998).

[0032] In this invention, we construct non-separable Shannon-like GFMRAs of $L^2(\mathbb{R}^n)$ whose scaling functions are radial and are defined with respect to certain unitary systems, which we will later introduce. We also derive certain of their associated frame multiwavelet sets. Our construction is the first of its kind. Scaling functions that are radial have not been constructed in the past. However, certain classes of non separable scaling functions in two dimensions, with some continuity properties with respect to dyadic dilations or dilations induced by the Quincunx matrix only have been constructed in the past (*e.g.*, ~~[7, 12, 11 8], [4]~~ A. Cohen and I. Daubechies. Nonseparable Bidimensional Wavelet Bases. *Revista Matematica Iberoamericana,* 9:51-137, 1993; J. Kovacevic and M. Vetterli. Nonseparable Multidimensional Perfect Reconstruction Filter-banks. *IEEE Transactions on Information Theory,* 38:533-555, 1992; W. He and M.J. Lai. Examples of Bivariate Nonseparable Compactly Supported Orthonormal Continuous Wavelets. In M. Unser, A. Aldroubi, A. Laine editor, *Wavelet Applications in Signal and Image Processing IV,* volume 3169 of *Proceedings SPIE,* pages 303-314, 1997; K. Grochenig and W. Madych. Multiresolution Analysis, Haar Bases and Self-Similar Tilings. *IEEE Transactions on Information Theory,* 38:558-568, 1992; and A. Ayache, E. Belogay, and Y. Wang. Orthogonal Lifting: Constructing New (Symmetric) Orthogonal Scaling Functions. 2002.). All of them have no axial symmetries and are not smooth, except those constructed in ~~[5]~~ E. Belogay and Y. Wang. Arbitrarily Smooth Orthogonal Nonseparable Wavelets in r^2. *SIAM Journal of Mathematical Analysis,* 30:678-697, 1999, which can be made arbitrarily smooth, but are highly asymmetric. Another construction in the spirit of digital filter design, but not directly related to wavelets can be found in ~~[1]~~ E.H. Adelson, E. Simoncelli, and R. Hingoranp. Orthogonal Pyramid Transforms for Image Coding. In *Visual*

Communications and Image Processing II, Volume 845 of *Proceedings SPIE*, pages 50-58, 1987 and ~~[18]~~ E.P. Simoncelli, W.T. Freeman, E.H. Adelson, and J. P. Hager. Shiftable Multi-Scale Transforms. *IEEE Transactions Information Theory*, 38(2):587-607, 1992. The latter construction and this of curvelets (*e.g.*, see ~~[19]~~ J. Starck, E. J. Candes, and D.L. Donoho. The Curvelet Transform for Image Denoising. *IEEE Transactions Image Processing*, 11(6):670-684, 2002) share two properties of our Radial GFMRAs: the separability of the designed filters with respect to polar coordinates and the redundancy of the induced representations. However, our construction in contrast to those due to Simoncelli et. al. and to Starck et al. are in the spirit of classical multiresolution analysis and can be carried out to any number of dimensions and with respect to a variety of dilation matrices.

[0033] The merit of non separable wavelets and scaling functions is that the resulting processing of images is more compatible with that of human or mammalian vision, because mammals do not process images vertically and horizontally as separable filter banks resulting from separable multiresolution analyses do (~~[20]~~ M. Vetterli and J. Kovacevic. *Wavelets and Subband Coding.* Prentice Hall PTR, Englewood Cliffs, NJ, 1995). As Marr suggests in his book ~~[13]~~ D. Marr. *Vision, A Computational Investigation into the Human Representation and Processing of Visual Information.* W. H. Freeman and Co., New York, N.Y., 1982, the human visual system critically depends on edge detection. In order to model this detection, Marr and Hildreth used the Laplacian operator, which is a "lowest order isotropic operator" (~~[14]~~ D. Marr and E. Hildreth. The Theory of Edge Detection. *Proc. R. Soc. London B*, 207:187-217, 1980), because our visual system is orientation insensitive to edge detection. Thus, the most desirable property in filter design for image processing is the isotropy of the filter. Radial scaling functions for multiresolutions based on frames are the best (and, according to proposition 5, the only) type of image processing filters that meet the isotropy requirement.

[0041] There exists a mapping $\sigma : \boldsymbol{G} \to \boldsymbol{G}$ satisfying

$$\boldsymbol{gD} = \boldsymbol{D}\sigma(\boldsymbol{g}), \quad \text{for every } \boldsymbol{g} \in \boldsymbol{G}$$

This particular assumption implies that σ is an injective homomorphism and $\sigma(\boldsymbol{G})$ is a subgroup of $\boldsymbol{G}$. (See ~~[9]~~ D. Han, D.R. Larson, M. Papadakis, and T. Stavropoulos. Multiresolution Analysis of Abstract Hilbert Spaces and Wandering Subspaces. In D. R. Larson L. Baggett, editor, *The Functional and Harmonic Analysis of Wavelets and Frames,*

volume 247 of *Cont. Math.*, pages 259-284. *Amer. Math. Soc.*, 1999 for proofs)

$|G : \sigma(G)| = n < +\infty$, where $|G : \sigma(G)|$ is the index of the subgroup $\sigma(G)$.

[0062] The frame scaling function can be determined in terms of Bessel functions, because it is a radial function.

$$\hat{\phi}(R) = \frac{J_{n/2}(\pi R)}{(2R)^{n/2}}, \quad R > 0$$

The proof of equation (7) can be found in ~~[17]~~ M. Pinsky. *An Introduction to Fourier Analysis and Wavelets.* 2001 Lemma 2.5.1.

[0063] We will not give any details regarding Bessel functions. However, the reader may refer to ~~[17]~~ M. Pinsky. *An Introduction to Fourier Analysis and Wavelets.* 2001 and ~~[3]~~ G.E. Andrews, R. Askey, and R Roy. *Special Functions.* Number 71 in Encyclopedia of Mathematics. Cambridge University Press, 2000 for an extensive treatment of their main properties and of course to the bible of the topic ~~[21]~~ G.N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge Mathematical Library. Cambridge University Press, 1944. Here, we only include the following formula.

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+a}}{k!\,\Gamma(k+a+1)}, \quad a > -1,\ x > 0$$

The function, J_a, given by the above equation is called the Bessel function of the first kind of order a.

[0078] Using a theorem relating to the radial FMRA $\{V_j\}_j$, we first set $\hat{V}_j = F(V_j)$ and $\hat{W}_j = F(W_j)$, where $j \in \mathbb{Z}$ $\hat{V}_0 = F(V_0) = L^2(\mathrm{D})$, and that i a unitary operator on $L^2(\mathbb{R}^n)$. Combining these facts with $\hat{V}_{-1} = L^2(B^{-1}(\mathrm{D}))$, we concl

$$\hat{W}_{-1} = \hat{V}_0 \cap \hat{V}_{-1}^{\perp} = L^2(Q)$$

where Q is the annulus $\mathrm{D} \cap (B^{-1}(\mathrm{D}))^c$, and the superscript c denotes the set-theoretic complement. Since an arbitrary orthogonal projection R defined on a Hilbert space H maps every orthonormal basis of H onto a Parseval frame for $R(H)$ (~~[2, 10]~~ A. Aldroubi. Portraits of Frames. *Proceeding of the American Mathematical Society*, 123: 1661-1668, 1995, and D. Han and D.R. Larson. *Frames, Bases and Group Representations*, volume 147 of *Memoirs.* American Mathematical Society, 2000),

we obtain that the orthogonal projection defined on $L^2(\mathbf{T}^n)$ by multiplication with the indicator function of Q gives a Parseval frame for $L^2(Q)$, namely the set $\{e_k\chi_Q : \mathbf{k} \in Z^n\}$.

[0082] One of the instrumental tools of this construction is the square root of the autocorrelation function Φ, which is defined by $A(\xi)^2 = \Phi(\xi)$, *a.e.* on $\mathbf{T}^n$. Also, the inverse of $A(\xi)$ is defined on the range of $\Phi(\xi)$ and is denoted by $A(\xi)^{-1}$. It can also been proved that the range projection P of the Analysis operator S is defined by $P\omega(\xi) = P(\xi)\omega(\xi)$, where $\omega \in L^2(\mathbf{T}^n)$, and that for *a.e.* $\xi \in \mathbf{T}^2$ the range projection of $\Phi(\xi)$ is the projection $P(\xi)$. For the sake of completeness, it must be noted that $P(\cdot)$ is a projection-valued weakly measurable function defined on $\mathbf{T}^2$. Since $\Phi = \chi_{\mathbf{D}}$, we deduce $P(\xi) = \chi_{\mathbf{D}}(\xi)$ a.e. in $\mathbf{T}^2$. The latter observation in conjunction with the preceding argument implies that $A(\xi)^{-1} = 1$, if $\xi \in \mathrm{D}$. For all other $\xi \in \mathbf{T}^2$, we have $A(\xi) = 0$, so for these ξ, we adopt the notational convention $A(\xi)^{-1} = 0$. Last but not least, an abelian group very instrumental in the discussion that follows is the kernel of the homomorphism ρ defined by

$$\rho(\xi)(\mathbf{k}) = e^{2\pi(\xi A\mathbf{k})}, \quad \mathbf{k} \in \mathbf{Z}^n$$

The latter equation implies that, for every $\xi \in \mathbf{T}^2$, $\rho(\xi)$ is the unique point in $\mathbf{T}^2$, such that $\rho(\xi) + \mathbf{k} = A^T\xi$. The kernel of ρ is homeomorphically isomorphic to a dual group of the quotient group $Z^2/A(Z^2)$ as shown in greater detail in reference ~~[16]~~ M. Papadakis. Generalized Frame Multiresolution Analysis of Abstract Hilbert Spaces. 2001. Now, let us fix $\mathbf{k}_r$, where $r = 0, 1, \dots, p - 1$, in $\mathbf{T}^2$, so that $\mathrm{Ker}\rho = \{\mathbf{k}_r : r = 0, 1, \dots, p - 1\}$.

~~REFERENCES~~

~~[0099] The following references are have been cited in the specification above:~~

1. ~~E.H. Adelson, E. Simoncelli, and R. Hingoranp. Orthogonal Pyramid Transforms for Image Coding. In *Visual Communications and Image Processing II*, Volume 845 of *Proceedings SPIE*, pages 50-58, 1987.~~
2. ~~A. Aldroubi. Portraits of Frames. *Proceeding of the American Mathematical Society*, 123: 1661-1668, 1995.~~
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4. ~~A. Ayache, E. Belogay, and Y. Wang. Orthogonal Lifting: Constructing New (Symmetric) Orthogonal Scaling Functions. 2002. Personal Communication.~~
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